RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2603(A)

Pure Mathematics 3

Section A

Monday **12 JANUARY 2004** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer **all** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The total number of marks for this paper is 60.

NOTE

- This paper will be followed by **Section B: Comprehension.**

This question paper consists of 4 printed pages.

HN/3

[Turn over

1 **(a)** Given that $x = t^3$ and $y = \sqrt{1+t}$, find $\frac{dy}{dx}$ in terms of t. [4]

(b) Find the first four terms in ascending powers of x of the binomial expansion of $\frac{1}{(1+2x)^3}$.

State the range of values of x for which the expansion is valid. [5]

(c) Express $\tan\left(x + \frac{\pi}{4}\right)$ in terms of $\tan x$.

Hence show that $\tan\left(x + \frac{\pi}{4}\right)\tan\left(x - \frac{\pi}{4}\right) = -1$. [4]

(d) Using small angle approximations for sine and cosine, show that, for small values of x,

$$\frac{1 - \cos x}{x \sin 2x} \approx \frac{1}{4}.$$ [3]

[Total 16]

2 **(i)** Given that $\frac{1}{x(1+x)^2} \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$, find A, B and C. [5]

(ii) Hence show by integration that the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x(1+x)^2}$$

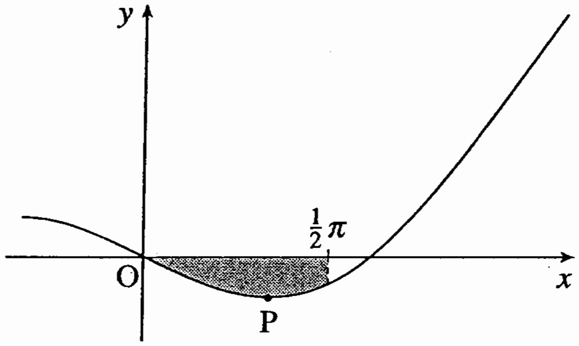
has the general solution

$$\ln y = \ln\left(\frac{x}{1+x}\right) + \frac{1}{1+x} + c.$$ [5]

(iii) Given that $y = \frac{1}{2}$ when $x = 1$, find the exact value of c. Find also the value of y when $x = 2$, giving your answer to 3 significant figures. [4]

[Total 14]

3 Fig. 3 shows part of the curve $y = x - 2\sin x$. P is a minimum point on the curve.



Not to scale

Fig. 3

(i) Show that the x-coordinate of P is $\frac{\pi}{3}$. [3]

(ii) Evaluate $\int_0^{\frac{1}{2}\pi} x\sin x\,dx$. [4]

(iii) Show that $\int_0^{\frac{1}{2}\pi} \sin^2 x\,dx = \frac{\pi}{4}$. [4]

(iv) The shaded region bounded by the curve, the x-axis and the line $x = \frac{1}{2}\pi$ is rotated through 360° about the x-axis. Using your results from parts **(ii)** and **(iii)**, find the volume of the solid of revolution formed, giving your answer in terms of π. [4]

[Total 15]

4 Fig. 4 shows a tetrahedron ABCD with vertices A(1, 0, 0), B(0, 1, 0), C(0, 0, 2) and D(−2, 0, 0).

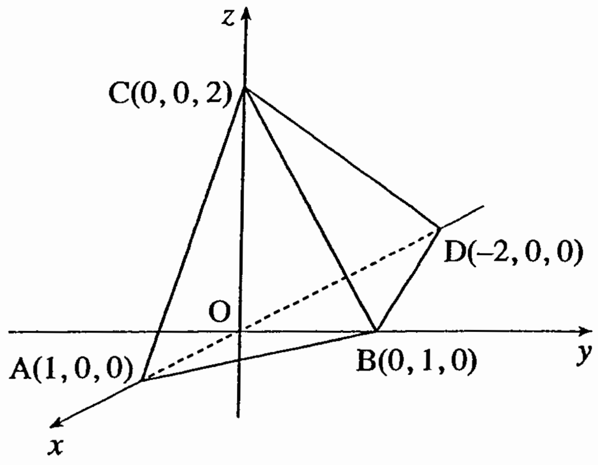


Fig. 4

(i) Find the scalar product $\overrightarrow{BA}.\overrightarrow{BC}$. Hence or otherwise find the angle ABC. [4]

(ii) Verify that the equation of the plane ABC is $2x + 2y + z = 2$. Write down a vector normal to the plane ABC. [4]

(iii) Write down a vector equation of the perpendicular l from the point D to the plane ABC. [2]

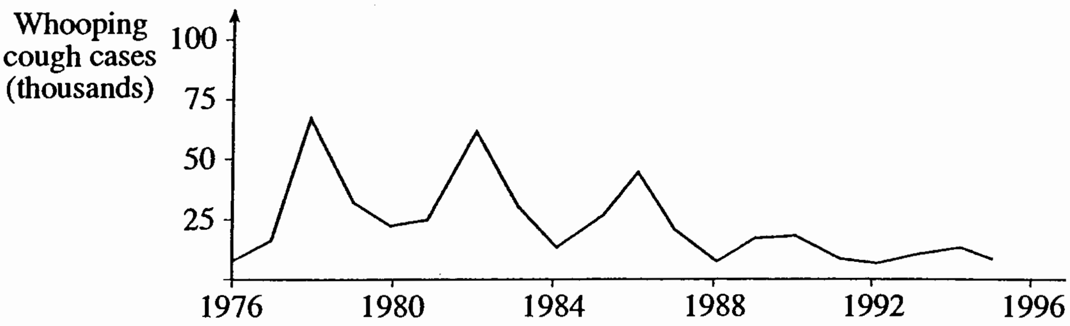
(iv) By finding where l meets the plane, find the distance from D to the plane ABC. [5]

[Total 15]

Herd Immunity

Introduction

The graph in Fig. 1 shows the number of cases of whooping cough in England and Wales from 1976 to 1995. A major feature of this graph is its 4-year cyclic pattern. There are minima in 1976, 1980, 1984, 1988 and 1992, and maxima midway between them, in 1978, 1982, 1986, 1990 and 1994.



Source: Medical Microbiology p314 **Fig.1**

This article explains why such a cyclic pattern occurs, and considers the implications for mass vaccination programmes in the human population.

Modelling the spread of a disease

Whooping cough is by no means the only disease to show a regular cyclic pattern. The same sort of pattern occurs in many animals with a variety of diseases and lengths of period.

To understand the process involved, it is helpful to construct a simple model. A population of animals is divided into three categories: susceptible, infected and immune.

At birth an individual animal is susceptible. During its lifetime it may, or may not, contract the disease. If it does contract the disease it becomes infected. Once it has recovered from the disease it becomes immune and so will not contract it again.

This is illustrated in Fig. 2. The population has size N; the numbers of animals in the three groups are S, F and I and so $S + F + I = N$. The proportions in the three categories are denoted by s, f and i; $s + f + i = 1$.

For the sake of simplicity, it is assumed that the disease is not fatal, so that all infected animals eventually recover and become immune.

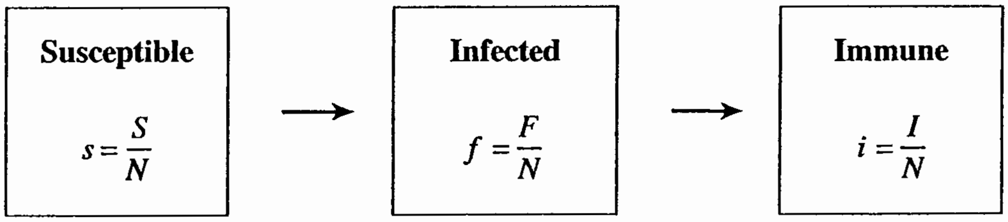


Fig. 2

The disease is caused by an organism (e.g. a virus or a bacterium). In order for the disease to spread beyond an infected animal, the organism must pass to other members of the population. While an animal is infected, it is also infectious; at this time the organism can pass to other animals in the population.

Some organisms are highly infectious (for example the foot and mouth virus) but many others actually find it quite difficult to spread. During the period when an animal is infectious, it will have sufficiently close contact with a number of other animals for the organism to pass on. The average number of such *contacts* is called the *basic reproductive rate* and it is denoted by R_0. (Notice that it is the organism that is reproducing and not the infected animal.)

If the infected animal is a member of a population which is otherwise completely susceptible, on average R_0 animals will then contract the disease.

If, on the other hand, some members of the population are already infected or immune, the number contracting the disease is, on average, less than R_0 because not all of the contacts are susceptible.

As an example, think of a population which is 30% susceptible ($s = 0.3$) and a disease for which $R_0 = 10$. On average an infected animal has 10 contacts. Of these, 3 contract the disease on average.

The average number of contacts contracting the disease is called the *effective reproductive rate* and it is denoted by R.

$$R = sR_0$$

In the example above $R = 3$. One infected animal passes the disease on to 3 more. In turn each of those passes it on to 3 more and so on, and the disease spreads through the population. This is described as an *epidemic*.

By contrast think of the situation when an animal is infected with the same disease but in a population for which $R = 0.5$. In that case, on average, less than 1 animal will be infected and contract the disease. Consequently the disease will die out.

In the critical case when $R = 1$, the disease neither dies out nor increases.

During an epidemic, the value of s decreases as susceptible animals contract the disease. The value of i increases as they recover from it and become immune. Eventually the value of R becomes less than 1, and the disease dies out.

Herd immunity

Think of a population in which the disease is not present and so $f = 0$. Some of the animals are susceptible and the rest are immune. Then one animal contracts the disease (say from a contact outside the population).

What happens next depends on the value of R. If it is greater than 1, the disease spreads and there is an epidemic.

By contrast, if the value of R is less than 1, the disease does not take hold in the population; instead it dies out. In this situation the population is said to have *herd immunity*.

[Turn over

For a population with no infected animals

$$s + i = 1,$$

and so herd immunity occurs when

$$(1 - i)R_0 < 1.$$

In the previous example, the value of R_0 was 10, and so herd immunity occurs if $i > 0.9$, that is if 90% or more of the population are immune.

Thus a population can have herd immunity without every individual animal being immune.

An iterative model

The process illustrated in Fig. 2 can be used to construct an iterative model for the spread of a disease.

Suppose that at a certain time, t, a number, F_t, of animals are infected. While an animal is infected, it passes the disease on to R more animals, giving a total number of new cases of $R \times F_t$.

The unit of time is taken to be the interval during which an animal is infected (i.e. the time from contracting the disease to recovering from it). It follows that one time unit later, at time $t + 1$, the number of infected animals is given by

$$F_{t+1} = R \times F_t.$$

This may be written as

$$F_{t+1} = R_0 s_t \times F_t$$

or as

$$F_{t+1} = R_0 \frac{S_t}{N} \times F_t$$

where s_t and S_t are the proportion and number of susceptible animals at time t.

The equations for the numbers of susceptible and immune animals may be written, using equivalent notation, as follows.

Immune animals: $I_{t+1} = I_t + F_t$

Susceptible animals: $S_{t+1} = S_t - F_{t+1}$

It is now possible to run this iteration on a spreadsheet. This is illustrated in Table 3 for the case where a single animal in a population of size 10 000 contracts a disease and where R_0 has the value 2.5. It is assumed that, at the start, 1000 animals in the population are immune.

Notice that the spreadsheet calculations have been carried out with unrounded numbers but, for ease of reading, values have been rounded to the nearest integer. As a consequence, in some rows of the table the total number of animals is not 10 000. The values of the effective reproductive rate, R, are given to 2 decimal places.

Time t	Susceptible S_t	Infected F_t	Immune I_t	R
0	8999	1	1000	2.25
1	8997	2	1001	2.25
2	8992	5	1003	2.25
3	8980	11	1008	2.25
4	8955	26	1020	2.24
5	8898	57	1045	2.22
6	8770	127	1102	2.19
7	8492	279	1230	2.12
8	7900	592	1508	1.97
9	6731	1169	2100	1.68
10	4764	1967	3269	1.19
11	2421	2343	5236	0.61
12	1003	1418	7579	0.25
13	648	356	8997	0.16
14	590	58	9352	0.15
15	582	8	9410	0.15
16	580	1	9418	0.15
17	580	0	9420	0.15

Table 3

At time $t = 11$, the value of R falls below 1 and so herd immunity has been acquired; the disease then starts to die out. However a few susceptible animals never contract the disease; in this case there are 580 of them, just under 6% of the population.

Modelling assumptions

Although the iterative model above shows the main features of the spread of the disease, it is actually quite crude.

It is based on the unit of time being the interval for which an animal is infected. Thus an animal passes from Susceptible to Infected to Immune in successive times. A better model would involve shorter time intervals but this would make it more complicated, with the infected group divided into several smaller groups.

There are also modelling assumptions of a more biological nature. For example, no allowance has been made for the possibility of some animals acquiring immunity from a parent without ever contracting the disease.

Cyclic epidemics

The ideas in this model can be extended to the phenomenon of cyclic epidemics, as illustrated for whooping cough in humans in Fig. 1.

Imagine a stable population of 1000 animals which are subject to a disease for which $R_0 = 2$. Each animal lives for 5 years. So every year 200 animals die of old age and are replaced by the same number of babies; the babies are all susceptible.

In the year 2000 there is an epidemic at the end of which the whole population is taken to be immune. (This is a modelling assumption to keep the numbers simple. As has already been seen, it would not normally be quite the case in practice.) The value of R can then be tracked over subsequent years, as shown in Table 4. (In Table 4 the animals' ages are given in completed years.)

	2000		2001		2002		2003	
Age	S	I	S	I	S	I	S	I
0	0	200	200	0	200	0	200	0
1	0	200	0	200	200	0	200	0
2	0	200	0	200	0	200	200	0
3	0	200	0	200	0	200	0	200
4	0	200	0	200	0	200	0	200
Total	0	1000	200	800	400	600	600	400
$R = SR_0/N$	0		0.4		0.8		1.2	

Table 4

The last row of the table shows that the value of R is less than 1 for three years but in 2003 it rises above 1, and an epidemic will occur. So, in this example, the disease has a 3-year cycle.

Vaccination programmes

So far it has been assumed that the only way for an animal to acquire immunity is by contracting the disease. This is not always so. For some diseases immunity can be passed from parent to offspring, and for others the same effect is produced by vaccination.

Table 4 was constructed for animals with a 5-year life span. They could be farm animals, and in that case it is realistic to think in terms of some of them being vaccinated at birth. In Table 5, the same animals are considered, with the same disease, but this time it is assumed that 40% of the animals, i.e. 80 animals, are vaccinated at birth and so are immune.

	2000		2001		2002		2003		2004		2005		
Age	*S*	*I*	*S*	*I*	*S*	*I*	*S*	*I*	*S*	*I*	*S*	*I*	**Age**
0	0	200	120	80	120	80	120	80	120	80	120	80	**0**
1	0	200	0	200	120	80	120	80	120	80	120	80	**1**
2	0	200	0	200	0	200	120	80	120	80	120	80	**2**
3	0	200	0	200	0	200	0	200	120	80	120	80	**3**
4	0	200	0	200	0	200	0	200	0	200	120	80	**4**
Total	0	1000	120	880	240	760	360	640	480	520	600	400	**Total**
$R = SR_0/N$	0		0.24		0.48		0.72		0.96		1.2		$R = SR_0/N$

Table 5 (40% vaccination at birth)

As a result of the vaccination, the value of R does not rise above 1 until 2005. So the period of the cycle has increased from 3 to 5 years.

Human public health programmes

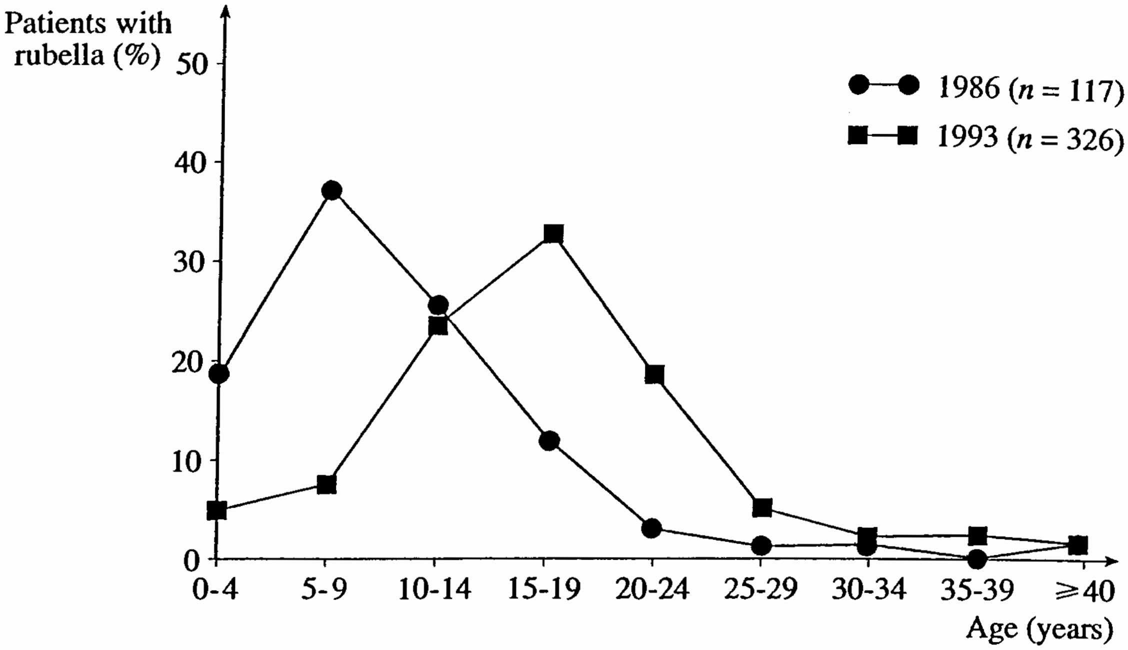
The above example illustrates the fact that unless the rate of vaccination is at or above that needed to give herd immunity, epidemics will continue to occur but less frequently.

This point is crucially important for human public health programmes. Many of the common human diseases, although usually fairly mild in children, can be extremely serious if caught by adults. The so-called "childhood diseases" of chicken pox, measles, mumps and rubella all come within this category.

Before the days of vaccination, epidemics of these diseases were common. Many parents would take their children to someone who was infected hoping that the children would contract the disease. That way they would acquire immunity at an age when the disease could be expected to be relatively mild. (However, there was a risk attached to this since even in childhood these diseases could prove fatal or result in disability.)

The effect of a public vaccination programme that fails to achieve herd immunity is thus that more people get the disease later in life. The serious consequences of this are illustrated by the example of vaccination against rubella in Greece. If a pregnant woman contracts rubella, the unborn child can suffer major disability or die.

Vaccination against rubella was first available in Greece in 1975 but uptake remained at a low level (under 50%) until 1990. There were epidemics in 1986 and in 1993. Those affected by the 1986 epidemic were mostly children, but in 1993 it was mostly adolescents and young adults, including many pregnant women. The age profiles for these two epidemics are illustrated in Fig. 6. The total numbers affected are also given.



Data source: British Medical Journal 4/12/99 **Fig. 6**

The proportion of pregnant women in Greece who were susceptible to rubella rose from about 12% before vaccination to 36% in 1990. From a public health perspective, a half-hearted programme of vaccination is much worse than having no vaccination at all. In the U.K. the government has set doctors targets that will ensure herd immunity, with financial rewards for meeting them; however this has created its own problems with some parents suspicious of doctors' impartiality in seeking to vaccinate their children.

For Examiner's Use

1 Use Fig. 6 to estimate the number of 15–19 year olds who contracted rubella in the 1993 outbreak. [1]

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2 Derive the result in page 4 line 5 that, for herd immunity, $i > 0.9$. [2]

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3 A population of animals has size 6000. The members of the population can contract a disease for which the basic reproductive rate is 15. The population does not have herd immunity. What can you say about the number of animals that are susceptible? [4]

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For Examiner's Use

4 Table 5 tracked a disease for which $R_0 = 2$ on the assumption of 40% vaccination. The period of the epidemic cycle was 5 years.

By completing the table below, find the period of the epidemic cycle when the vaccination rate is reduced to 30%. The value of R_0 remains 2. [4]

	2000		2001		2002		2003		2004		2005		
Age	S	I	S	I	S	I	S	I	S	I	S	I	Age
0	0	200											0
1	0	200											1
2	0	200											2
3	0	200											3
4	0	200											4
Total	0	1000											Total
$R = SR_0/N$	0												$R = SR_0/N$

The period of the epidemic cycle is .... years.

5 On page 4, an iterative model is introduced. Here are two of the three equations.

$$I_{t+1} = I_t + F_t$$

$$S_{t+1} = S_t - F_{t+1}$$

Explain the meanings of these equations in words. [4]

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SECTION B

1 About 33% of 326 = 108	B1 [1]	Accept 104 – 111 Accept eg 107.58
2 $s + i = 1 \Rightarrow s = 1 - i \Rightarrow R = sR_0 = (1-i)R_0 < 1$ for herd immunity. If $R_0 = 10$ $(1-i)10 < 1$ $\Rightarrow$ $(1-i) < 0.1$ $\Rightarrow$ $i > 1 - 0.1 = 0.9$	M1 E1 [2]	Use of $R < 1$ and $R_0 = 10$ www
3 For herd immunity $sR_0 < 1$ So in this case $sR_0 \geq 1$ $R_0 = 15 \therefore 15s \geq 1 \Rightarrow s \geq \frac{1}{15} \Rightarrow \frac{S}{N} \geq \frac{1}{15}$ $N = 6000 \therefore \frac{S}{6000} \geq \frac{1}{15} \Rightarrow S \geq \frac{6000}{15} \therefore S \geq 400$	M1 M1 A1 A1 [4]	Substituting $R_0 = 15$ into sR_0 or $(1-i)R_0$ <,=, or > 1 Using $S = 6000s$ or $I = 6000i$ soi $S = 400$ or $I = 5600$ soi $S \geq 400$ Accept >

4

	20	00	20	01	20	02	20	03
Age	***S***	***I***	***S***	***I***	***S***	***I***	***S***	***I***
0	0	200	140	60	140	60	140	60
1	0	200	0	2300	140	60	140	60
2	0	200	0	200	0	200	140	60
3	0	200	0	200	0	200	0	200
4	0	200	0	200	0	200	0	200
Total	0	1000	140	860	280	720	420	580
R=SR₀/N	0		0.2	8	0.5	6	0.8	4

B1 — 140 and 60 seen

	20	04	20	05
Age	***S***	***I***	***S***	***I***
0	140	60		
1	140	60		
2	140	60		
3	140	60		
4	0	200		
Total	560	440		
R=SR₀/N	1.1	2		

M1 — Establishing a pattern with 140 and 60
A1 — All the values of R correct

The period of the epidemic cycle is 4 years — B1ft [4] — Cycle length. ft their values of R.

5 Immune animals $I_{t+1} = I_t + F_t$ In the next time interval the immunes will consist of those who are currently immune and those who are currently infected. Susceptible animals $S_{t+1} = S_t - F_{t+1}$ In the next time interval, the susceptibles will consist of those who were previously susceptible less those among them that have just got infected.	B1 B1 B1 B1 [4] Total[15]	Accept 'at time t+1', 'at time t', 'initially', 'the following time period', 'after one time period', etc. And allow the Bs as long as the time period for the term in question is clear.

Mark Scheme

1 (a) $x = t^3 \Rightarrow \frac{dx}{dt} = 3t^2$ $y = \sqrt{1+t} \Rightarrow \frac{dy}{dt} = \frac{1}{2}(1+t)^{-1/2}$ $\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ $= \frac{\frac{1}{2}(1+t)^{-1/2}}{3t^2}$ $= \frac{1}{6t^2\sqrt{1+t}}$	B1 B1 M1 A1 [4]	 $\frac{\text{their } dy/dt}{\text{their } dx/dt}$ soi Any correct form of the answer in which the 3 and the two have been combined as 6. Allow negative indices.
(b) $\frac{1}{(1+2x)^3} = (1+2x)^{-3}$ $= 1 + (-3)(2x) + \frac{(-3)(-4)}{2!}(2x)^2 + \frac{(-3)(-4)(-5)}{3!}(2x)^3 + \ldots$ $= 1 - 6x + 24x^2 - 80x^3 + \ldots$ Valid for $-1 < 2x < 1 \Rightarrow -½ < x < ½$ or $\lvert x \rvert < 1/2$	M1 B1 B1 B1 B1 [5]	All the binomial coeffs correct $-6x$ $+24x^2$ $-80x^3$ $\leq$ is B0
(c) $\tan(x + \frac{\pi}{4}) = \frac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}$ $= \frac{\tan x + 1}{1 - \tan x}$ $\tan(x + \frac{\pi}{4})\tan(x - \frac{\pi}{4}) = \frac{\tan x + 1}{1 - \tan x} \cdot \frac{\tan x - 1}{1 + \tan x}$ $= -1$*	M1 A1 M1 E1 [4]	Correct formula for tan $(A + B)$ used Allow retrospectively tan $(A - B)$ formula used www
(d) $\frac{1 - \cos x}{x \sin 2x} \approx \frac{1 - (1 - \frac{x^2}{2})}{x.2x}$ $= \frac{\frac{x^2}{2}}{2x^2}$ $= 1/4$*	M1 M1 E1 [3] [Total 16]	$\cos x \approx 1 - x^2/2$ used $\sin 2x \approx 2x$ or $\sin x \approx x$ used Condone the absence of brackets if the subsequent working is correct.

2 (i) $\frac{1}{x(1+x)^2} \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$ $\Rightarrow 1 \equiv A(1+x)^2 + Bx(1+x) + Cx$ $x = 0 \Rightarrow 1 = A$ $x = -1 \Rightarrow 1 = -C \Rightarrow C = -1$ coeff of x^2: $0 = A + B \Rightarrow B = -1$	M1 M1 A1 A1 A1 [5]	Correct identity. Any correct substitution or correct equating of coefficients or any equivalent valid method. All the coefficients correct implies both Ms $A = 1$ The accuracy marks are $B = -1$ dependent on the first M1 $C = -1$
(ii) $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x(1+x)^2}$ $\Rightarrow \int \frac{1}{y} dy = \int \frac{1}{x(1+x)^2} dx$ $= \int [\frac{1}{x} - \frac{1}{1+x} - \frac{1}{(1+x)^2}] dx$ $\Rightarrow \ln y = \ln x - \ln(1+x) + \frac{1}{1+x} + c$ $= \ln\frac{x}{1+x} + \frac{1}{1+x} + c$ *	M1 M1 B1ft B1ft E1 [5]	Separating the variables. Condone poor notation providing that the intention is clear. substituting their partial fractions $\ln x - \ln(1+x)$ $\frac{1}{1+x}$ condone the absence of c follow thro' their integral providing that the M1s have been earned. www. $\ln y$ and c must be seen
(iii) substituting $x = 1$ and $y = ½$: $\Rightarrow \ln ½ = \ln ½ + ½ + c$ $\Rightarrow c = -½$ when $x = 2$ $\ln y = \ln 2/3 + 1/3 - ½$ $\Rightarrow y = 0.564$ (3 s.f.)	M1 A1 M1 A1 [4] [Total 14]	Allow both M marks if the candidate uses his own equation providing the work is comparable cao

3 (i) $y = x - 2\sin x$ $\Rightarrow \frac{dy}{dx} = 1 - 2\cos x$ $= 0$ $\Rightarrow \cos x = ½$ $\Rightarrow x = \pi/3$*	B1 M1 E1 [3]	$\frac{dy}{dx} = 1 - 2\cos x$ their $\frac{dy}{dx} = 0$ $\cos x = ½$ must be seen, but allow verification
(ii) $\int_0^{\frac{\pi}{2}} x\sin x\,dx$ Let $u = x$, $dv/dx = \sin x$ $\Rightarrow v = -\cos x$ $= [-x\cos x]_0^{\pi/2} + \int_0^{\frac{\pi}{2}} \cos x.1dx$ $= 0 + [\sin x]_0^{\pi/2}$ $= 1$.	M1 A1 A1 A1 [4]	Correct choice of parts and an attempt at $[-x\cos x] + \int$ ..... $-x\cos x$ cao $\sin x$ cao cao
(iii) $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ $\cos 2x = 1 - 2\sin^2 x$ $\Rightarrow \sin^2 x = ½(1 - \cos 2x)$ $= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 2x)\,dx$ $= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2}$ $= \pi/4$*	M1 A1 A1ft E1 [4]	A reasonable attempt to use $\cos 2x$ $\int 1/2(1-\cos 2x)dx$ $\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]$ www
(iv) $V = \int_0^{\pi/2} \pi y^2 dx$ $= \int_0^{\pi/2} \pi(x - 2\sin x)^2 dx$ $= \pi\int_0^{\pi/2}(x^2 - 4x\sin x + 4\sin^2 x)dx$ $= \pi\left[\frac{x^3}{3}\right]_0^{\pi/2} - 4\pi + 4\pi.\frac{\pi}{4}$ $= \frac{\pi^4}{24} - 4\pi + \pi^2$	M1 B1 M1 A1 [4] Total [15]	$\int_0^{\pi/2} \pi y^2 dx$ Correct limits and π seen Correct expansion of the bracket $\left[\frac{x^3}{3}\right]$ + their result from (ii) and the above result from (iii) substituted. or equivalent.

4(i) $\vec{BA}=\begin{pmatrix}1\\0\\0\end{pmatrix}-\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\vec{BC}=\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\-1\\2\end{pmatrix}$ $\Rightarrow \vec{BA}.\vec{BC}=1\times0+(-1)\times(-1)+0\times2$ $=1$ $\cos ABC=\dfrac{\vec{BA}.\vec{BC}}{\lvert BA\rvert\lvert BC\rvert}=\dfrac{1}{\sqrt{2}.\sqrt{5}}$ $\Rightarrow ABC=71.6°$	M1 A1 M1 A1[4]	Allow any two vectors Accept omission of working or correct scalar product seen in the formula for cos ABC
(ii) $2x+2y+z=2$ At A (1,0,0) $2\times1+2\times0+0=2$ at B (0,1,0) $2\times0+2\times1+0=2$ at C (0,0,2) $2\times0+2\times0+2=2$ $\Rightarrow$ Equation of the plane ABC is $2x+2y+z=2$ Normal vector is $\begin{pmatrix}2\\2\\1\end{pmatrix}$ Alternative schemes <u>Vector equation :</u> M1 for the correct form of a vector equation of the plane. M1 for the complete elimination of two parameters. A1 for deducing the correct equation of the plane B1 for the normal vector.	M1 B2,1.0 B1 [4]	Substituting the coordinates of one point into the eq'n. Accept eg 2+0+0 = 2 but not 2=2. $k\begin{pmatrix}2\\2\\1\end{pmatrix}$ Accept a row vector, but not the equation of a particular normal <u>The use of a vector product</u>: M1 A1 for the correct vector product of two vectors in the plane. B1 for identifying the normal A1 for the RHS of the equation <u>The use of scalar products:</u> B1 for the normal vector. M1 for the scalar product of the normal and one vector in the plane. A1 for two such products = 0 A1 for the RHS of the equation
(iii) $\mathbf{r}=\begin{pmatrix}-2\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$	B1 B1ft [2]	$\begin{pmatrix}-2\\0\\0\end{pmatrix}+\ldots$ $\ldots+\lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$ Condone the omission of $\mathbf{r}=$
(iv) $\mathbf{r}=\begin{pmatrix}-2+2\lambda\\2\lambda\\\lambda\end{pmatrix}$: $2(-2+2\lambda)+4\lambda+\lambda=2$ $\Rightarrow 9\lambda=6$ $\Rightarrow \lambda=2/3$ So line meets plane at $(-\frac{2}{3},\frac{4}{3},\frac{2}{3})$. Distance = $\sqrt{(-2+\frac{2}{3})^2+(-\frac{4}{3})^2+(-\frac{2}{3})^2}=2$	M1 A1 A1ft M1 A1 [5]	ft their equation in (iii) ft their λ and their equation Or $\left\lvert\dfrac{2\times(-2)+2\times0+1\times0-2}{\sqrt{2^2+2^2+1^2}}\right\rvert=2$

Examiner’s Report

2603 Pure Mathematics 3

General Comments

There was a very good response to this paper with a high proportion of candidates scoring very good marks in the 60+ range. Of particular note was the number of candidates who obtained full marks on both sections A and B. There was a small number of candidates, scoring less than 20 marks, who perhaps were not yet ready to take this unit. The presentation of work was generally good and the work of many of the high scoring candidates was immaculate. Weaker candidates continue to suffer from poor notation, particularly in integration and work with vectors.

The response to the Section B comprehension was good. The questions that required arithmetic or algebra to answer them were very well answered, but where candidates needed to express their understanding of parts of the passage some were unable to do so adequately.

There was no evidence that any candidates were short of time in either section, nor were there many misinterpretations of questions.

Comments on Individual Questions

Question 1 (Various topics)

Most candidates attempted most of this question and, on average, it was the highest scoring question on the paper.

(a) A few candidates attempted to eliminate the parameter t and then find dy/dx directly in terms of x, but most made some error along the way and few returned to t. The great majority proceeded in the accepted way by differentiating with respect to t. The resulting negative index in dy/dt caused a problem for a few, eg $\frac{\frac{1}{2}(1+t)^{-\frac{1}{2}}}{3t^2} = \frac{\frac{1}{2\sqrt{1+t}}}{3t^2} = \frac{3t^2}{2\sqrt{1+t}}$, but perhaps the most common loss of a mark was in not combining the 2 and the 3 in the denominator of the result.

(b) The binomial expansion was very well done, just a few candidates confusing $(2x)^2$ with $2x^2$; but many candidates were unable to write down the correct range of validity, and $\leq$ was often seen.

(c) Most candidates were able to obtain the correct expressions for $\tan(x + \pi/4)$ and $\tan(x - \pi/4)$, although a few failed to make the substitution $\tan \pi/4 = 1$. However, a number of candidates spoiled their solution by clearly incorrect cancelling.

(d) Here again although most candidates knew the approximations for $\sin x$ and $\cos x$, some candidates lost the final mark because of incorrect simplification. One such error was $1 - (1 - \frac{x^2}{2}) = 1 - 1 - \frac{x^2}{2}$ or even $\frac{1-1-x^2}{2}$. Quite a large number of candidates used $\sin 2x = 2\sin x \cos x$ and needed to disregard x^4.

(a) $\frac{1}{6t^2\sqrt{1+t}}$; (b) $1 - 6x + 24x^2 - 80x^3 + \ldots$ Valid for $-\frac{1}{2} < x < \frac{1}{2}$

Question 2 (Partial fractions and differential equations)

Part (i) The work here was generally sound and many candidates obtained full marks. However a significant number gave the wrong identity, most commonly

$$1 \equiv A(1+x)(1+x)^2 + Bx(1+x)^2 + Cx(1+x),$$

but also, $1 \equiv A(1+x^2) + Bx(1+x) + Cx$.

Part (ii) Again there were many good, completely correct solutions showing full details of the separation of the variables, the substitution of the partial fractions and the integration. However many candidates gave the impression that they had worked backwards from the given solution; ln y appeared without any $\int \frac{1}{y}\mathrm{d}y$, and integral signs were placed in front of the partial fractions without any dx or other indication as to why they should be integrated. The final fraction $\frac{1}{1+x}$ was quite often seen in the solution without any further explanation even, sometimes, when there was an error in the sign of the corresponding partial fraction.

Part (iii) was well done even by many weaker candidates. There were occasional sign errors, such as $\ln \frac{1}{2} = \ln \frac{1}{2} + \frac{1}{2} + c \Rightarrow c = \frac{1}{2}$, but a more common error was in the next line when substituting the value $x = 2$, $\ln y = \ln \frac{2}{3} + \frac{1}{3} - \frac{1}{2} \Rightarrow y = \frac{2}{3} + e^{\frac{1}{3}} - e^{\frac{1}{2}}$.

(i) $A = 1$, $B = -1$, $C = -1$; (iii) 0.564

Question 3 (Integration)

Part (i) This was very well answered by almost all candidates. Just an occasional candidate gave $\frac{\mathrm{d}y}{\mathrm{d}x}$ as $1 + 2\cos x$. A small number of candidates verified the result, which was acceptable.

Part (ii) Also well done. Again just a few candidates had a sign error, this time integrating sin$x$ to cos$x$.

Part (iii) This was not well answered except by the most able candidates. Some had sign errors in the use of the double angle formula; some, starting from $\cos 2x = 1 - 2\sin^2 x$, had algebraic errors in obtaining $\sin^2 x$ and some attempted integration by substitution obtaining $\frac{1}{3}\sin^3 x$.

Part (iv) Only a few good candidates were able to complete this. Candidates either did not know the formula for the volume of a solid of revolution, or made an error in squaring y or omitted the π.

(ii) 1; (iv) $\frac{\pi^4}{24} - 4\pi + \pi^2$

Question 4 (Vectors)

This question was mostly well answered.

Part (i) A few candidates got **BA** or **BC** or both the wrong way round, and occasionally notation was poor, **BA** . **BC** = $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ = 1.

Part (ii) A variety of methods was used in this part, most of them correctly. Writing down a vector equation of the plane ABC and eliminating the two parameters was very popular. In this case the elimination was quite easy and therefore nearly always correctly done. The straightforward substitution of the coordinates of points A, B and C was also done correctly by many. A variation of this using scalar products, eg $\begin{pmatrix}1\\0\\0\end{pmatrix}.\begin{pmatrix}2\\2\\1\end{pmatrix}=2$, was not always successful because it was not repeated for the other two points. Similarly a few candidates showing that $\mathbf{BA}.\begin{pmatrix}2\\2\\1\end{pmatrix}=0$ and therefore BA is perpendicular to the normal to the plane, failed to repeat the process for **BC** or **CA**.

Some further mathematics candidates used the vector product of, eg **BA** and **BC**, to find the normal to the plane and then calculated *d*. Occasionally such candidates failed to identify the normal as such.

Part (iii) was an easy write down for most candidates.

Part (iv) Well done by many candidates. Having obtained the point of intersection of the perpendicular from D to the plane some candidates found the distance of this point from the origin instead of the length of the perpendicular. Others made an error in subtracting the coordinates of this point from D.

(i) 71.6°; (ii) $\begin{pmatrix}2\\2\\1\end{pmatrix}$; (iii) $\mathbf{r}=\begin{pmatrix}-2\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$; (iv) 2

Section B (Comprehension)

Q1 Almost all candidates read the graph correctly but some failed to change the percentage of patients with rubella into the *number* of 15-19 year olds.

Q2 This question was usually well answered. Just a few candidates were not convincing with their argument for the final inequality.

Q3 This proved to be the most difficult question. A significant number of candidates failed to understand that if the population of animals does not have herd immunity there is a minimum number of animals which are susceptible to the disease. These candidates wrote generally about how the number of susceptible animals changes over the course of an epidemic. However, those candidates who started, correctly, from the statement that for herd immunity $sR_0 < 1$ usually made some progress. A few of these candidates referred back to question 2 and used $i > 0.9$, and a few confused s and i. Others reached the correct conclusion that $s \geq 1/15$ and then failed to convert from the proportion of the population to the number of animals.

Q4 was very well answered indeed, almost all candidates scoring the four marks available.

Q5 This question required accurate use of language and quite a large number of candidates were not up to it. Also, it was clear that many candidates were thinking in terms of just two periods instead of a number of periods in an iterative model. Nevertheless there were many good answers.

1. 108 (104-111); 3. $S \geq 400$; 4. 4 years